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THE EFFECT OF THE WALLS OF AN ARBITRARY TANK IN THE PROBLEM OF SEPARATION-FREE IMPACT ON A FLOATING BODY

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An algorithm for constructing an asymptotic power series for large depths is proposed. It allows one to use the well-known solution of the problem of impact on a rigid body floating on the surface of a fluid half-space to obtain an approximate solution of the impact problem for the same body floating on the surface of a fluid in a bounded basin. The case where the domain occupied by the fluid has two perpendicular planes of symmetry is considered. Asymptotic expressions are given for the velocity potential on the wetted part of the body surface and for the added mass. Examples of solutions are considered.

In a previous paper [1], we proposed an algorithm for constructing an asymptotic power series for large depths that allows one to use the well-known solution of the problem of impact on a rigid body immersed in a fluid half-space to obtain an approximate solution of the impact problem for the same body immersed in a fluid layer of finite depth. In the present paper, this algorithm is extended to the case of an arbitrary bounded basin. The results obtained are also valid for certain unbounded domains (layer, semiinfinite cylinder, etc.).

The proposed asymptotic approach is based on Stokes's classical method of successive approximations.the same time, in computational mathematics there is Schwarz's method. In both methods, solution of the original problem for a geometrically complex domain reduces to successive solution of problems for domains having a simpler boundary.

In the present paper, we describe an algorithm for constructing an asymptotic series for the case of central impact on a floating body.

1. Formulation of the Problem. We consider a rigid body floating on the surface of an ideal incompressible fluid occupying a bounded basin of arbitrary shape. Before impact, the body and the fluid were at rest. As a result of impact, the body begins to move in the vertical direction without rotation (central impact). We assume that the shape of the body is such that there is no separation of the fluid from the wetted body surface (separation-free impact). For a separation-free impact, it suffices to require that the normal components of the velocities of points on the body surface be nonnegative everywhere on the wetted body surface. The boundary of the domain occupied by the fluid is considered piecewise-smooth.

The potential of the velocities acquired by the fluid particles as a result of the impact is denoted by $V_0\Phi$, where the dimensionless potential Φ is determined by solving the following mixed problem of the potential theory for the domain occupied by the fluid [2, 3]:

$$\Delta\Phi = 0, \qquad \frac{\partial\Phi}{\partial n}\Big|_{S_1} = n_z, \qquad \Phi\Big|_{S_2} = 0, \qquad \frac{\partial\Phi}{\partial n}\Big|_{S_3} = 0.$$

Here S_1, S_2, and S_3 are the wetted surface of the rigid body, the free surface of the fluid, and the fixed rigid wall of the basin, respectively, V_0 is the velocity acquired by the body upon the impact, and n_z is the

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projection of the outward normal to the surface S_1 onto the axis z. Cartesian coordinates x, y, and z are introduced so that the x and y axes lie in the plane of the free surface, the z axis is directed into the fluid, and the coordinate origin coincides with a certain point of the body.

We assume that homothety with center at the origin and coefficient h maps the fixed surface S_3^0 onto the boundary S_3: $S_3 = hS_3^0$ ($x = hx^0$, $y = hy^0$, and $z = hz^0$). The fluid density ρ is set equal to unity.

The following notation is used: G is the infinite domain bounded by the wetted body surface S_1 and the free fluid surface S_2 (the case of $h = \infty$), D is the domain bounded by the surface S_3 and the plane $z = 0$, D^0 is the inner domain bounded by the surface S_3^0 and the plane $z = 0$, and ∂D (∂D^0) is the union of the surface S_3 (S_3^0) and its mirror reflection about the plane $z = 0$.

2. Construction of an Asymptotic Solution for Large *h*. The proposed method involves successive solution of the following two problems: the case of $h = \infty$ (boundary-value problem in the domain G) and the problem in a bounded basin with no body (boundary-value problem in the domain D). In both cases, we eliminate the residuals that arise on the fixed boundary S_3 and the wetted surface of the body S_1. Expanding the approximations obtained in a power series in h^{-1} and retaining the necessary number of terms, we obtain an asymptotic formula for large h.

Let us discuss the problem in detail. We seek the velocity potential Φ in the form of the series $\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \ldots$. As the first approximation Φ_1, we use the solution of the impact problem for the body floating on the surface of a fluid half-space. At large distances from the body, the following Fourier-series expansion of the potential Φ_1 holds [3, 4]:

$$\Phi_1 = -\frac{C_1 z}{2\pi R^3} - \frac{C_2 xz + C_3 yz}{4\pi R^5} - \frac{C_4 z^3 + C_5 x^2 z + C_6 y^2 z + C_7 xyz}{R^7} - \ldots \,. \tag{2.1}$$

Here $R = \sqrt{x^2 + y^2 + z^2}$, the constants $C_1, C_2, \ldots, C_7$ are expressed in terms of integrals over the wetted surface of the body, involving the velocity potential Φ_1, e.g.,

$$C_1 = \iint\limits_{S_1} z \frac{\partial \Phi_1}{\partial n}\, ds - \iint\limits_{S_1} \Phi_1 n_z\, ds, \qquad \frac{C_2}{6} = \iint\limits_{S_1} xz \frac{\partial \Phi_1}{\partial n}\, ds - \iint\limits_{S_1} zn_x \Phi_1\, ds - \iint\limits_{S_1} xn_z \Phi_1\, ds,$$

$$\frac{C_3}{6} = \iint\limits_{S_1} yz \frac{\partial \Phi_1}{\partial n}\, ds - \iint\limits_{S_1} zn_y \Phi_1\, ds - \iint\limits_{S_1} yn_z \Phi_1\, ds.$$

To eliminate the residuals caused by the potential Φ_1 on the fixed boundary S_3, we consider the problem in a bounded basin with no body:

$$\Delta \Phi_2 = 0, \quad \Phi_2\Big|_{z=0} = 0, \quad \frac{\partial \Phi_2}{\partial n}\Big|_{S_3} = \frac{C_1}{2\pi} Q_1\Big|_{S_3} + Q_2\Big|_{S_3},$$
$$Q_1 = \frac{\partial}{\partial n} \frac{z}{R^3}, \quad Q_2 = \frac{\partial}{\partial n} \frac{C_2 xz + C_3 yz}{4\pi R^5}. \tag{2.2}$$

Here it suffices to consider only two terms of series (2.1). The contribution of the remaining terms to the potential Φ on the wetted surface of the body is of the order of $O(h^{-5})$ for $h \to \infty$. After odd continuation of the function Φ_2 across the plane $z = 0$, the solution of problem (2.2) is represented as the sum of the simple- and double-layer potentials:

$$\Phi_2(x_0, y_0, z_0) = \frac{1}{4\pi} \iint\limits_{\partial D} \frac{1}{R_{pop}} \frac{\partial \Phi_2}{\partial n}\, ds - \frac{1}{4\pi} \iint\limits_{\partial D} \Phi_2 \frac{\partial}{\partial n} \frac{1}{R_{pop}}\, ds,$$
$$P = (x, y, z), \quad P_0 = (x_0, y_0, z_0), \quad R_{pop} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}. \tag{2.3}$$

In the surface integrals in (2.3), we change the variables: $x \to hx$, $y \to hy$, $z \to hz$, and $ds \to h^2 ds$. As a result, we obtain the following representation for the potential Φ_2:

$$\Phi_2(x_0,y_0,z_0)=\frac{1}{4\pi}\iint\limits_{\partial D^0}\frac{1}{R_h}\Big(\frac{C_1}{2\pi h^2}\,\frac{\partial f}{\partial n}+\frac{1}{h^3}\,\frac{\partial g}{\partial n}\Big)ds-\frac{1}{4\pi}\iint\limits_{\partial D^0}\Big(\frac{C_1 f}{2\pi h^2}+\frac{g}{h^3}\Big)\frac{\partial}{\partial n}\,\frac{1}{R_h}\,ds,$$

$$R_h=\sqrt{(x_0/h-x)^2+(y_0/h-y)^2+(z_0/h-z)^2}.$$

Here the functions f and g are defined as the solutions of the following boundary-value problems in the domain D^0:

$$\Delta f=0,\quad f\Big|_{z=0}=0,\quad \frac{\partial f}{\partial n}\Big|_{S_3^0}=Q_1\Big|_{S_3^0},\qquad \Delta g=0,\quad g\Big|_{z=0}=0,\quad \frac{\partial g}{\partial n}\Big|_{S_3^0}=Q_2\Big|_{S_3^0}. \tag{2.4}$$

Expanding Φ_2 as a function of the parameter $\varepsilon=1/h$ in a Taylor series about the point $\varepsilon=0$ $(h=\infty)$, we obtain the asymptotic formula

$$\Phi_2(x_0,y_0,z_0)=-\frac{C_1\xi}{2\pi}\,z_0h^{-3}-(\xi_1z_0x_0+\xi_2z_0y_0+\xi_3z_0)h^{-4}+O(h^{-5})$$

$$\Big(\xi=\frac{1}{2\pi}\iint\limits_{S_3^0}\Big(f\,\frac{\partial f_1}{\partial n}-f_1\,\frac{\partial f}{\partial n}\Big)ds,\qquad f_1=\frac{z}{R^3},\qquad \xi_1,\,\xi_2,\,\xi_3=\mathrm{const}\Big), \tag{2.5}$$

which are valid in any fixed (independent of h) neighborhood of the wetted surface of the body $(h\to\infty)$.

To compensate for the normal components of the potential Φ_2 that arose on the wetted surface of the body, we consider the case of $h=\infty$ again. Ignoring the remainder term in Eq. (2.5), we obtain the following boundary-value problem in the domain G for Φ_3:

$$\Delta\Phi_3=0,\qquad \Phi_3\Big|_{S_2}=0,\qquad \Phi_3\Big|_{\infty}=0,$$

$$\frac{\partial\Phi_3}{\partial n}\Big|_{S_1}=\frac{C_1\xi}{2\pi}\,n_zh^{-3}+[z(\xi_1n_x+\xi_2n_y)+(\xi_1x+\xi_2y+\xi_3)n_z]h^{-4}.$$

According to the last boundary condition, the function Φ_3 is represented as the sum of two terms, one of which differs from Φ_1 by a constant factor: $\Phi_3=(C_1\xi/(2\pi))h^{-3}\Phi_1+h^{-4}\Phi_*$.

Collecting the approximations constructed, we obtain the following asymptotic formula for the potential Φ on the wetted body surface:

$$\Phi=\Phi_1+\frac{C_1\xi}{2\pi}\,h^{-3}(\Phi_1-z_0)+O(h^{-4})\qquad (h\to\infty). \tag{2.6}$$

If the potential Φ_2 is determined taking into account not only the first two terms of series (2.1) but also the other terms, the process of successive approximations can be continued by successively considering the problem in a bounded basin with no body and the case of $h=\infty$ (at next step, the residuals caused by the potential Φ_3 on the fixed boundary S_3 are eliminated, and the procedure is then repeated). However, the next approximations add only terms of the order of $O(h^{-6})$ to the asymptotic formula of the potential Φ (2.6) for $h\to\infty$. Hence, the first two terms of the harmonic series (2.1) allow us to write the asymptotic formula for the potential Φ on the wetted surface of the body up to terms of the order of $O(h^{-5})$ for $h\to\infty$. We do not give the expression for the third term in the asymptotic formula (2.6) because of its awkwardness.

The constant ξ can be written in a simpler form. Applying Green's formula to the functions f and f_1 in the domain obtained from the domain D^0 by elimination of a hemisphere S_ε of small radius ε with center at the coordinate origin, we obtain

$$\xi=\frac{1}{2\pi}\iint\limits_{S_\varepsilon}\Big(f\,\frac{\partial f_1}{\partial n}-f_1\,\frac{\partial f}{\partial n}\Big)ds. \tag{2.7}$$

On the hemisphere S_ε, we have $f_1 = z/\varepsilon^3$ and $\partial f_1/\partial n = -2z/\varepsilon^4$. Converting now to spherical coordinates in (2.7) and letting the parameter ε tend to zero, we obtain

$$\xi = -\frac{\partial f}{\partial z}\Big|_{M_0}, \qquad M_0 = (0, 0, 0). \tag{2.8}$$

Thus, to find the constant ξ, we need to solve the boundary-value problem (2.4) and then to calculate the derivative (2.8). The constant ξ is independent of the geometry of the floating body and depends only on the shape of the boundary S_3.

We note that if the surface S_3^0 (or S_3) has two perpendicular symmetry planes xz and yz, the remainder term in Eq. (2.6) is of the order of $O(h^{-5})$ for $h \to \infty$. This can be explained by the fact that the coefficients ξ_1, ξ_2, and ξ_3 at h^{-4}, are expressed in terms of integrals of odd (in x or y) functions over S_3^0. In a particular case where S_3^0 is a surface of revolution, it is convenient to introduce a stream function ψ related to f by the formulas

$$\frac{\partial f}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad \frac{\partial f}{\partial z} = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{2.9}$$

where the function ψ is defined as the solution of the following boundary-value problem in the domain D^0 $[\psi(0) = 0]$:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0, \qquad \frac{\partial \psi}{\partial z}\Big|_{z=0} = 0, \qquad \psi\Big|_{S_3^0} = \frac{r^2}{(r^2 + z^2)^{3/2}}, \qquad r^2 = x^2 + y^2.$$

Using formula (2.6), one can easily find asymptotic expressions for the total impact momentum and its moment about the origin exerted on the body during the impact $(h \to \infty)$:

$$\boldsymbol{B} = \boldsymbol{B}_\infty + \frac{C_1\xi}{2\pi h^3}(\boldsymbol{B}_\infty - V_0\,\boldsymbol{L}_1) + O(h^{-4}), \qquad \boldsymbol{M} = \boldsymbol{M}_\infty + \frac{C_1\xi}{2\pi h^3}(\boldsymbol{M}_\infty - V_0\,\boldsymbol{L}_2) + O(h^{-4}),$$

$$\boldsymbol{L}_1 = (0, 0, V), \quad \boldsymbol{L}_2 = (L_{21}, L_{22}, 0), \quad L_{21} = \int\limits_V y\,dV, \quad L_{22} = -\int\limits_V x\,dV.$$

Here $\boldsymbol{B}_\infty$ and $\boldsymbol{M}_\infty$ are the momentum and the moment of momentum for $h = \infty$, respectively, and V is the volume of the submerged part of the body.

We now assume that the domain occupied by the fluid has two perpendicular symmetry planes xz and yz. In this case, the equations for variations of the momentum and the moment of momentum due to central impact yield the relations

$$(m_\text{b} + m)V_0 = P_z, \quad P_x = 0, \quad P_y = 0, \quad x_0 = \frac{m_\text{b} p_0}{m_\text{b} + m}, \quad y_0 = \frac{m_\text{b} q_0}{m_\text{b} + m},$$

where m_b is the mass of the body, m is the added mass, p_0 and q_0 the abscissa and ordinate of the center of mass of the body, P_x, P_y, and P_z are the components of the external impact momentum applied to the point with the coordinates x_0, y_0, and z_0.

Hence, in the case of the indicated symmetry, the body must be impacted by a vertical force applied at the point with the abscissa x_0 and the ordinate y_0.

The asymptotic expressions for the potential Φ at the wetted body surface and for the added mass m take the form

$$\begin{gathered} \Phi = \Phi_1 + \frac{(m_\infty + V)\xi}{2\pi h^3}(\Phi_1 - z_0) + O(h^{-5}) \qquad (h \to \infty), \\ m = m_\infty + \frac{(m_\infty + V)^2\xi}{2\pi h^3} + O(h^{-5}) \qquad (h \to \infty). \end{gathered} \tag{2.10}$$

Here m_∞ is the added mass for $h = \infty$.

3. Applications. We present the values of the constant ξ for a number of particular domains:

— for a fluid layer of finite depth, $\xi = 3\zeta(3)/(8a^3)$ [$\zeta(x)$ is the Riemann zeta function];

— for a semiinfinite cylinder,

$$\xi = \frac{2}{\pi a^3} \int\limits_0^\infty \frac{\lambda^2 K_1(\lambda)}{I_1(\lambda)}\, d\lambda$$

[$I_1(\lambda)$ and $K_1(\lambda)$ are modified Bessel functions of the first and the third kind];

— for a hemisphere, $\xi = 2/a^3$;

— for half-space with a circular outer barrier, $\xi = 2/(3\pi a^3)$.

Here a is the characteristic size of the domain D^0, which is the depth of the layer in the first example, the radius of the cylinder in the second example, and the radius of the hemisphere in the third example. In the last example, the surface S_3^0 is the exterior of a circle of radius a in the plane $z = 0$. In the first two examples, substitution of the constant ξ into Eq. (2.10) yields results coinciding with the results of [1].

Conclusions. In this paper, we derive simple asymptotic formulas that allow one to take into account the effect of the walls of an arbitrary basin in the problem of central impact on a floating body in the case of moderate depths.

By analogy, one can solve the problem of a vertical separation-free impact on a floating body in the case where the body starts to move in the vertical direction and rotate about the horizontal axis. If the domain occupied by the fluid has two perpendicular symmetry planes xz and yz, the second terms of the asymptotic potentials for rotations about the x and y axes are of the order of $O(h^{-5})$ for $h \to \infty$.

The methods for constructing asymptotics proposed in this paper can be generalized to the case where two arbitrary bodies interact with each other on the surface of a fluid half-space.

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